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Serre–Swan theorem for non-commutative C^*-algebras

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Abstract

For a Hilbert C^*-module X over a C^*-algebra $\mathcal{A}$, we introduce a vector bundle $\mathcal{E}_X$ associated to X. We prove that $\mathcal{E}_X$ has an hermitian metric and a flat connection. We introduce a vector space Γ_X of holomorphic sections of $\mathcal{E}_X$ with the following properties: (i) Γ_X is a Hilbert $\mathcal{A}$-module, (ii) the action of $\mathcal{A}$ on Γ_X is defined by means of the connection of $\mathcal{A}$, (iii) the C^*-inner product of Γ_X is induced by the hermitian metric of $\mathcal{E}_X$.

We prove that the Hilbert C^*-module Γ_X is isomorphic to X.

This sectional representation is a generalization of the Serre–Swan theorem to non-commutative C^*-algebras. We show that $\mathcal{E}_X$ is isomorphic to an associated bundle of an infinite dimensional Hopf bundle with the structure group $U(1)$.

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1. Introduction

The Serre–Swan theorem [8] is described as follows.

Theorem 1.1 (Serre–Swan). *Let Ω be a connected compact Hausdorff space and $C(\Omega)$ the algebra of all complex valued continuous functions on Ω. Assume that X is a module over*

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$C(\Omega)$. Then X is finitely generated projective iff there is a complex vector bundle E on Ω such that X is isomorphic onto the module of all continuous sections of E.

By Theorem 1.1, finitely generated projective modules over $C(\Omega)$ and complex vector bundles on Ω are in one-to-one correspondence up to isomorphisms. In non-commutative geometry [5,11], some class of modules over a non-commutative C^*-algebra $\mathcal{A}$ are treated as vector bundles on a "non-commutative space" $\mathcal{A}$, generalizing Serre–Swan theorem for commutative C^*-algebras.

On the other hand, for a unital general non-commutative C^*-algebra $\mathcal{A}$, there is a uniform Kähler bundle $(\mathcal{P}, p, B)$ [3] unique up to equivalence class of $\mathcal{A}$, such that $\mathcal{A}$ is * isomorphic onto a uniform Kähler function algebra on $(\mathcal{P}, p, B)$, what is a natural generalization of Gel'fand representation. We carefully review the uniform Kähler bundle and the functional representation of non-commutative C^*-algebras in Section 2. Under the above consideration, we state the following theorem which is a version of the Serre–Swan theorem generalized to non-commutative C^*-algebras.

Theorem 1.2. *Let X be a Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$, $(\mathcal{P}, p, B)$ the uniform Kähler bundle of $\mathcal{A}$, $\mathcal{K}_u(\mathcal{P})$ the C^*-algebra of uniform Kähler functions on $\mathcal{P}$ and $f : \mathcal{A} \cong \mathcal{K}_u(\mathcal{P})$ the Gel'fand representation of $\mathcal{A}$:*

(i) *There is a complex vector bundle $\mathcal{E}_X$ on $\mathcal{P}$ with a hermitian metric H and a flat connection D and a bundle map P_X from the trivial vector bundle $X \times \mathcal{P}$ on $\mathcal{P}$ to $\mathcal{E}_X$ with dense image, at each fiber.*

(ii) *Let $\Gamma_X \equiv (P_X)_*(\Gamma_{\mathrm{const}}(X \times \mathcal{P})) \subset \Gamma_{\mathrm{hol}}(\mathcal{E}_X)$, where $\Gamma_{\mathrm{const}}(X \times \mathcal{P})$ is the set of all constant sections of $X \times \mathcal{P}$ and $\Gamma_{\mathrm{hol}}(\mathcal{E}_X)$ is the set of all holomorphic sections of $\mathcal{E}_X$. Then Γ_X is a Hilbert $\mathcal{K}_u(\mathcal{P})$-module with the right $*$-action*

$$\Gamma_X \times \mathcal{K}_u(\mathcal{P}) \to \Gamma_X,$$
$$(s, l) \mapsto s * l \equiv s \cdot l + \sqrt{-1} D_{X_l} s \quad ((s, l) \in \Gamma_X \times \mathcal{K}_u(\mathcal{P})),$$

and the C^*-inner product

$$H|_{\Gamma_X \times \Gamma_X} : \Gamma_X \times \Gamma_X \to \mathcal{K}_u(\mathcal{P}),$$

where X_l is the holomorphic part of the complex Hamiltonian vector field of $l \in \mathcal{K}_u(\mathcal{P}) \subset C^\infty(\mathcal{P})$ with respect to the Kähler form of $\mathcal{P}$.

(iii) *Under an identification $f : \mathcal{A} \cong \mathcal{K}_u(\mathcal{P})$, Γ_X is isomorphic to X as a Hilbert $\mathcal{A}$-module.*

In Section 3.1, we introduce the atomic bundle $\mathcal{E}_X$ of a Hilbert C^*-module X, which is a Hilbert bundle on $\mathcal{P}$. We discuss its geometrical structure in Section 3.3. In Section 4.1, we define a flat connection D on the atomic bundle. In Section 4.2, we prove that any connection on the atomic bundle defines a $*$-action of the algebra of smooth functions on $\mathcal{P}$ on the vector space of holomorphic sections of $\mathcal{E}_X$. In Section 5, we give a proof of Theorem 1.2.

Here we summarize correspondences between geometry and algebra

Gel'fand representation

	space	algebra
CG	Ω	$C(\Omega)$ pointwise product
NCG	$\mathcal{P} \to B$	$\mathcal{K}_u(\mathcal{P})$ $*$-product

Serre-Swan theorem

	vector bundle	module
CG	$E \to \Omega$	$\Gamma(E)$ pointwise action
NCG	$\mathcal{E}_X \to \mathcal{P}$	Γ_X $*$-action

where we call respectively, CG = commutative geometry as a geometry associated with commutative C^*-algebras, and NCG = non-commutative geometry as a geometry associated with non-commutative C^*-algebras by following [4].

2. Preparation

2.1. Uniform Kähler bundles

We start from the geometric characterization of the set of all pure states and the spectrum of a C^*-algebra [3]. Assume now that E and M are topological spaces.

Definition 2.1. (E, μ, M) is called a uniform Kähler bundle if it satisfies the following conditions:

(i) μ is an open, continuous surjection between E and M,
(ii) the topology of E is induced by a given uniformity,
(iii) each fiber $E_m \equiv \mu^{-1}(m)$ is a Kähler manifold.

The local triviality of uniform Kähler bundle is not assumed. M is, in general, neither compact nor Hausdorff.

We simply denote (E, μ, M) by E. For uniform spaces, see [2]. Any metric space is a uniform space. Examples and relations with concrete C^*-algebras are given in Example 2.6. Roughly speaking, the fibers of the uniform Kähler bundle taken into account the non-commutativity of the C^*-algebra.

Definition 2.2. Two uniform Kähler bundles (E, μ, M), (E', μ', M') are isomorphic if there is a pair (β, ϕ) of a uniform homeomorphism $\beta : E \to E'$ and a homeomorphism $\phi : M \to M'$, such that $\mu' \circ \beta = \phi \circ \mu$ and any restriction $\beta|_{\mu^{-1}(m)} : \mu^{-1}(m) \to (\mu')^{-1}(\phi(m))$ is a holomorphic Kähler isometry for any $m \in M$. We call (β, ϕ) a uniform Kähler isomorphism between (E, μ, M) and (E', μ', M').

For example, any Kähler manifold N is a uniform Kähler bundle with a one-point set as the base space. In the same way, the metric direct sum of Kähler manifolds $\{N_i\}_{i=1}^n$ is a uniform Kähler bundle with a n-point set as base space, endowed with the discrete topology.

Any compact Hausdorff space X is a uniform space. X is a uniform Kähler bundle with zero-dimensional fiber with itself as the base space [2].

We explain the non-trivial third example of uniform Kähler bundles as follows. Let $\mathcal{A}$ a unital C^*-algebra. Denote $\mathcal{P}$ the set of all pure states of $\mathcal{A}$, endowed with the w^*-uniformity, i.e. the uniformity which induces the w^*-topology. By the GNS representation of $\mathcal{A}$, there is a natural projection p from $\mathcal{P}$ onto the spectrum B of $\mathcal{A}$, the set of all equivalence classes of irreducible representations of $\mathcal{A}$. The projection p is continuous when B is endowed with the Jacobson topology [10].

If $\mathcal{A}$ is commutative, then $\mathcal{P} \cong B \cong$ "the set of all maximal ideals of $\mathcal{A}$" is a compact Hausdorff space. In [3] the following results are proved.

Theorem 2.3 (Reduced atomic realization). *For any unital C^*-algebra $\mathcal{A}$, $(\mathcal{P}, p, B)$ is a uniform Kähler bundle.*

Let $(\mathcal{H}_b, \pi_b)$ be an irreducible representation belonging to $b \in B$. Then $\rho \in \mathcal{P}$ corresponds $[x_\rho] \in \mathcal{P}(\mathcal{H}_b) \equiv (\mathcal{H}_b \setminus \{0\})/\mathbf{C}^\times$, where $\rho = \omega_{x_\rho} \circ \pi_b$ and ω_{x_ρ} denotes a vector state $\omega_{x_\rho} = \langle x_\rho|(\cdot)x_\rho\rangle$. Then $\mathcal{P}_b$ has a Kähler manifold structure induced by the bijection

$$\tau^b : \mathcal{P}_b \to \mathcal{P}(\mathcal{H}_b), \quad \tau^b(\rho) \equiv [x_\rho]. \tag{2.1}$$

The Kähler distance d_b on a fiber $\mathcal{P}_b \equiv \mathcal{P}(\mathcal{H}_b)$ is given by

$$d_b(\rho, \rho') \equiv \sqrt{2}\,\mathrm{arcos}|\langle x_\rho|x_{\rho'}\rangle| \quad (\rho, \rho' \in \mathcal{P}_b),$$

which is the length of shortest geodesic arc between ρ and ρ' in $\mathcal{P}_b$.

Theorem 2.4. *Let $\mathcal{A}_i$ be C^*-algebras with associated uniform Kähler bundles $(\mathcal{P}_i, p_i, B_i)$, $i = 1, 2$. Then $\mathcal{A}_1$ and $\mathcal{A}_2$ are * isomorphic if and only if $(\mathcal{P}_1, p_1, B_1)$ and $(\mathcal{P}_2, p_2, B_2)$ are isomorphic as uniform Kähler bundles.*

By this theorem, the uniform Kähler bundle $(\mathcal{P}, p, B)$ associated with $\mathcal{A}$ is uniquely determined up to uniform Kähler isomorphisms. From now on, we call it *the uniform Kähler bundle associated with $\mathcal{A}$.*

2.2. A functional representation of non-commutative C^*-algebras

We reconstruct $\mathcal{A}$ from the uniform Kähler bundle $(\mathcal{P}, p, B)$ associated with $\mathcal{A}$. Since $\mathcal{P}_b \equiv p^{-1}(b) \subset \mathcal{P}$ is a Kähler manifold for each $b \in B$, we can consider the fiberwise smooth (= smooth in $\mathcal{P}_b$ for each $b \in B$) functions on $\mathcal{P}$. Let

$C^\infty(\mathcal{P})$: the set of all fiberwise smooth complex valued functions on $\mathcal{P}$.

For $l \in C^\infty(\mathcal{P})$, we denote X_l the holomorphic Hamiltonian vector field of l, defined by the equation

$$\omega_\rho((X_l)_\rho, \bar{Y}_\rho) = \bar{\partial}_\rho l(\bar{Y}_\rho) \quad (\bar{Y}_\rho \in \bar{T}_\rho\mathcal{P}) \quad \text{for } \rho \in \mathcal{P}, \tag{2.2}$$

where ω denotes the Kähler form on $\mathcal{P}$ (defined on each fiber), $\bar{\partial}$ the anti-holomorphic differential operator on $C^\infty(\mathcal{P})$ and $\bar{T}_\rho\mathcal{P}$ denotes the anti-holomorphic tangent space of $\mathcal{P}$ at $\rho \in \mathcal{P}$. A product $*$ on $C^\infty(\mathcal{P})$ is defined by

$$l * m \equiv l \cdot m + \sqrt{-1}X_m l \quad (l, m \in C^\infty(\mathcal{P})). \tag{2.3}$$

If the involution * is defined on $C^\infty(\mathcal{P})$ by complex conjugation, then $(C^\infty(\mathcal{P}), *)$ becomes a *-algebra with unit which is not associative in general. By using (2.2), the $*$-product can be written as follows:

$$l * m = l \cdot m + \sqrt{-1}\omega(\bar{X}_l, X_m).$$

Let us introduce the Kähler bracket $\{\cdot, \cdot\}$ with respect to ω, by

$$\{l, m\} \equiv \omega(\bar{X}_l, X_m) + \omega(X_l, \bar{X}_m) \quad (l, m \in C^\infty(\mathcal{P})).$$

Then the following equality holds:

$$l * m - m * l = \sqrt{-1}\{l, m\} \quad (l, m \in C^\infty(\mathcal{P})). \tag{2.4}$$

Theorem 2.5 (Gel'fand representation of non-commutative C^*-algebras). *For a non-commutative C^*-algebra $\mathcal{A}$, the Gel'fand representation*

$$f_A(\rho) \equiv \rho(A) \quad (A \in \mathcal{A}, \rho \in \mathcal{P}),$$

gives an injective * homomorphism of unital *-algebras:

$$f : \mathcal{A} \to C^\infty(\mathcal{P}), \quad A \mapsto f_A,$$

where $C^\infty(\mathcal{P})$ is endowed with the above defined $*$-product. For a function l in the image $f(\mathcal{A})$ of the map f, set

$$\|l\| \equiv \sup_{\rho\in\mathcal{P}} |(\bar{l} * l)(\rho)|^{1/2}. \tag{2.5}$$

This defines a C^*-norm on the associative *-subalgebra $f(\mathcal{A})$. By this norm, $(f(\mathcal{A}), *)$ is isomorphic to $\mathcal{A}$.

Furthermore $f(\mathcal{A})$ is precisely the subset $\mathcal{K}_u(\mathcal{P}) \subset C^\infty(\mathcal{P})$ defined by

$$\mathcal{K}_u(\mathcal{P}) \equiv \left\{ l \in C^\infty(\mathcal{P}) : \begin{array}{c} D^2 l = 0,\, \bar{D}^2 l = 0 \\ \bar{l} * l,\, l * \bar{l},\, l \text{ are uniformly continuous on } \mathcal{P} \end{array} \right\}, \tag{2.6}$$

where D and *$\bar{D}$ are the holomorphic and anti-holomorphic part, respectively, of covariant derivative of Kähler metric defined on each fiber of $\mathcal{P}$. Hence, the following equivalence of C^*-algebras holds*:

$$\mathcal{A} \cong \mathcal{K}_u(\mathcal{P}).$$

We call $(\mathcal{K}_u(\mathcal{P}), *)$ the C^*-algebra of uniform Kähler functions on $\mathcal{P}$.

By the above results, we obtain a fundamental correspondence between algebra and geometry as follows:

$$\begin{array}{lcl} \text{unital commutative } C^*\text{-algebra} & \Leftrightarrow & \text{compact Hausdorff space} \\ \bigcap & & \bigcap \\ \text{unital generally non-commutative } C^*\text{-algebra} & \Leftrightarrow & \text{uniform Kähler bundle associated with a } C^*\text{-algebra} \end{array}$$

The upper correspondence above is just the Gel'fand representation of unital commutative C^*-algebras.

Example 2.6. Assume that $\mathcal{H}$ is a separable infinite dimensional Hilbert space.

(i) When $\mathcal{A} \equiv \mathcal{L}(\mathcal{H})$ is the algebra of all bounded linear operators on $\mathcal{H}$, the uniform Kähler bundle of $\mathcal{A}$ is $(\mathcal{P}(\mathcal{H}) \cup \mathcal{P}_-, p, 2^{[0,1]} \cup \{b_0\})$, where $\mathcal{P}(\mathcal{H})$ is the projective Hilbert space of $\mathcal{H}$, $\mathcal{P}_-$ is the union of a family of projective Hilbert spaces indexed by the power set of the closed interval $[0, 1]$ and $\{b_0\}$ is the one-point set corresponding to the equivalence class of identity representation $(\mathcal{H}, \mathrm{id}_{\mathcal{L}(\mathcal{H})})$ of $\mathcal{L}(\mathcal{H})$ on $\mathcal{H}$. Since the primitive spectrum of $\mathcal{L}(\mathcal{H})$ is a two-point set, the topology of $2^{[0,1]} \cup \{b_0\}$ is equal to $\{\emptyset, 2^{[0,1]}, \{b_0\}, 2^{[0,1]} \cup \{b_0\}\}$ [7]. In this way, the base space of the uniform Kähler bundle is not always a singleton when the C^*-algebra is type I.

(ii) For the C^*-algebra $\mathcal{A}$ generated by the Weyl form of the one-dimensional canonical commutation relation $U(s)V(t) = e^{\sqrt{-1}st}V(t)U(s)$ for $s, t \in \mathbf{R}$, its uniform Kähler bundle is $(\mathcal{P}(\mathcal{H}), p, \{1pt\})$. The spectrum is a one-point set $\{1pt\}$ since von Neumann uniqueness theorem [1].

(iii) The *CAR-algebra* $\mathcal{A}$ is a UHF algebra with the nest $\{M_{2^n}(\mathbf{C})\}_{n\in\mathbf{N}}$. The uniform Kähler bundle has the base space $2^{\mathbf{N}}$ and each fiber on $2^{\mathbf{N}}$ is a separable infinite dimensional projective Hilbert space where $2^{\mathbf{N}}$ is the power set of the set $\mathbf{N}$ of all natural numbers with trivial topology, that is, the topology of $2^{\mathbf{N}}$ is just $\{\emptyset, 2^{\mathbf{N}}\}$. In general, the Jacobson topology of the spectrum of a simple C^*-algebra is trivial [7].

3. The atomic bundle of a Hilbert C^*-module

The aim of this section is the construction of a natural vector bundle for a given Hilbert C^*-module over a C^*-algebra.

3.1. *The construction of the atomic bundle*

Before starting to construct the atomic bundle of a Hilbert C^*-module, we state the definition of a Hilbert C^*-module.

Definition 3.1 ([6]). X is a Hilbert C^*-module over a C^*-algebra $\mathcal{A}$ if X is a right $\mathcal{A}$-module and there is an $\mathcal{A}$ valued sesquilinear form

$$\langle\cdot|\cdot\rangle : X \times X \to \mathcal{A},$$

which satisfies the following conditions:

$$\langle\eta|\xi a\rangle = \langle\eta|\xi\rangle a \quad (\eta, \xi \in X, a \in \mathcal{A}), \qquad (\langle\eta|\xi\rangle)^* = \langle\xi|\eta\rangle \quad (\eta, \xi \in X),$$
$$\langle\xi|\xi\rangle \geq 0 \quad (\xi \in X), \qquad \langle\xi|\xi\rangle = 0 \Rightarrow \xi = 0 \quad (\xi \in X),$$

and X is complete with respect to the $\mathcal{A}$-valued norm defined by

$$\|\xi\| \equiv \|\langle\xi|\xi\rangle\|^{1/2} \quad (\xi \in X). \tag{3.1}$$

Let X be a Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$ and $(\mathcal{P}, p, B)$ the uniform Kähler bundle associated with $\mathcal{A}$. Defining a closed subspace N_ρ of X with $\rho \in \mathcal{P}$ by

$$N_\rho \equiv \{\xi \in X : \rho(\|\xi\|^2) = 0\}, \tag{3.2}$$

we consider the quotient vector space

$$\mathcal{E}^o_{X,\rho} \equiv X/N_\rho, \tag{3.3}$$

equipped with the sesquilinear form $\langle\cdot|\cdot\rangle_\rho$ on $\mathcal{E}^o_{X,\rho}$ defined by

$$\langle\cdot|\cdot\rangle_\rho : \mathcal{E}^o_{X,\rho} \times \mathcal{E}^o_{X,\rho} \to \mathbf{C}, \qquad \langle[\xi]_\rho|[\eta]_\rho\rangle_\rho \equiv \rho(\langle\xi|\eta\rangle) \quad ([\xi]_\rho, [\eta]_\rho \in \mathcal{E}^o_{X,\rho}),$$

where

$$[\xi]_\rho \equiv \xi + N_\rho \in \mathcal{E}^o_{X,\rho} \quad (\xi \in X). \tag{3.4}$$

Then $\langle\cdot|\cdot\rangle_\rho$ becomes an inner product on $\mathcal{E}^o_{X,\rho}$. Let $\mathcal{E}_{X,\rho}$ be the completion of $\mathcal{E}^o_{X,\rho}$ by the norm $\|\cdot\|_\rho \equiv (\langle\cdot|\cdot\rangle_\rho)^{1/2}$. We obtain a Hilbert space $(\mathcal{E}_{X,\rho}, \langle\cdot|\cdot\rangle_\rho)$ from a Hilbert C^*-module X for each pure state $\rho \in \mathcal{P}$. We note that $\mathcal{E}_{X_\rho}$ and $\mathcal{E}_{X_{\rho'}}$ are equivalent Hilbert spaces when $\rho, \rho' \in \mathcal{P}_b$.

Definition 3.2. The atomic bundle $\mathcal{E}_X = (\mathcal{E}_X, \Pi_X, \mathcal{P})$ of a Hilbert C^*-module X over a C^*-algebra $\mathcal{A}$ is defined as the fiber bundle $\mathcal{E}_X$ on $\mathcal{P}$:

$$\mathcal{E}_X \equiv \bigcup_{\rho\in\mathcal{P}} \mathcal{E}_{X,\rho},$$

where the projection map $\Pi_X : \mathcal{E}_X \to \mathcal{P}$ is defined by $\Pi_X(x) = \rho$ for $x \in \mathcal{E}_{X,\rho}$.

The atomic bundle is the collection of its B-fibers, where for $b \in B$, the B-fiber $\mathcal{E}^b_X$ of X is the bundle $(\mathcal{E}^b_X, \Pi^b_X, \mathcal{P}_b)$, defined by

$$\mathcal{E}^b_X \equiv \bigcup_{\rho\in\mathcal{P}_b} \mathcal{E}_{X,\rho}, \qquad \Pi^b_X : \mathcal{E}^b_X \to \mathcal{P}_b, \quad \Pi^b_X \equiv \Pi_X|_{\mathcal{E}^b_X}.$$

3.2. *Unitary group action on the atomic bundle*

Let G be the group of all unitary elements in $\mathcal{A}$. Define an action χ of G on $\mathcal{P}$ by

$$\chi_u(\rho) \equiv \rho \circ \mathrm{Ad}\, u^* \quad (u \in G, \rho \in \mathcal{P}).$$

Then χ_u maps $\mathcal{P}_b$ to $\mathcal{P}_b$ for each $b \in B$ and $u \in G$.

Lemma 3.3. *G acts transitively on $\mathcal{P}_b$ by Kähler automorphisms.*

Proof. By irreducibility of the GNS representation of pure states, the statement follows immediately. □

Next, define an action t^b of G on $\mathcal{E}^o_X = \bigcup_{\rho\in\mathcal{P}} \mathcal{E}^o_{X,\rho}$ by

$$t^b_u([\xi]_\rho) \equiv [\xi u^*]_{\chi_u(\rho)} \quad (u \in G, [\xi]_\rho \in \mathcal{E}^o_{X,\rho}),$$

t^b is well defined since the map $\xi \mapsto \xi u^*$ maps N_ρ to $N_{\chi_u(\rho)}$. As t^b_u is a unitary map from $\mathcal{E}^o_{X,\rho}$ to $\mathcal{E}^o_{X,\chi_u(\rho)}$, we can extend t^b_u as a unitary map from $\mathcal{E}_{X,\rho}$ to $\mathcal{E}_{X,\chi_u(\rho)}$. We note that

$$t^b_{cu}(x) = \bar{c}t^b_u(x) \quad (u \in G, c \in U(1)). \tag{3.5}$$

We define an action t of G on $\mathcal{E}_X$ by $t|_{\mathcal{E}^b_X} \equiv t^b$, $b \in B$. Then $T \equiv (t, \chi)$ is an action of G on $(\mathcal{E}_X, \Pi_X, \mathcal{P})$ by bundle automorphisms. This action preserves B-fibers $(\mathcal{E}^b_X, \Pi^b_X, \mathcal{P}_b)$, $b \in B$, too.

Consider now the Hopf bundle $(S(\mathcal{H}_b), \mu_b, \mathcal{P}_b)$ (see Appendix A). For the fibrations $(\mathcal{E}^b_X, \Pi^b_X, \mathcal{P}_b)$ and $(S(\mathcal{H}_b), \mu_b, \mathcal{P}_b)$, define their fiber product $\mathcal{E}^{b,U(1)}_X \subset \mathcal{E}^b_X \times S(\mathcal{H}_b)$ by

$$\mathcal{E}^{b,U(1)}_X \equiv \mathcal{E}^b_X \times_{\mathcal{P}_b} S(\mathcal{H}_b) = \{(x, h) \in \mathcal{E}^b_X \times S(\mathcal{H}_b) : \Pi^b_X(x) = \mu_b(h)\}.$$

Thus an action σ^b of G on $\mathcal{E}^{b,U(1)}_X$ is defined by

$$\sigma^b_u(x, h) \equiv (t_u(x), \pi_b(u)h) \quad ((x, h) \in \mathcal{E}^{b,U(1)}_X, u \in G).$$

We note that a representation $(\mathcal{H}_b, \pi_b)$ of $\mathcal{A}$ induces an action of G on $S(\mathcal{H}_b)$.

Lemma 3.4. *For $(x, h) \in \mathcal{E}^{b,U(1)}_X$ and $u \in G$, if $\sigma^b_u(x, h) = (y, h)$, then $x = y$.*

Proof. We have just to consider the case $x \in \mathcal{E}^o_{X,\rho}$. Let $x = [\xi]_\rho$. By assumption, $(y, h) = ([\xi u^*]_{\chi_u(\rho)}, \pi_b(u)h)$. Hence $h = \pi_b(u)h$ or, equivalently

$$\pi_b(u^*)h = h. \tag{3.6}$$

By definition of fiber product, we have $\chi_u(\rho) = \rho$ and $y = [\xi u^*]_\rho$. By using the above results, we obtain:

$$\|x - y\|^2_\rho = \rho(\|\xi - \xi u^*\|^2) = \rho(\|\xi\|^2) + \rho(u\|\xi\|^2u^*) - \rho(\langle\xi|\xi\rangle u^*) - \rho(u\langle\xi|\xi\rangle),$$

with $\rho = \langle h|\pi_b(\cdot)h\rangle$. Therefore (3.6) implies

$$\|x - y\|^2_\rho = 2\rho(\|\xi\|^2) - \rho(\langle\xi|\xi\rangle) - \rho(\langle\xi|\xi\rangle) = 0.$$

Hence we obtain $x = y$. □

Definition 3.5. F^b_X is the set of all orbits of G in $\mathcal{E}^{b,U(1)}_X$.

Let $\mathcal{O}(x, h) \in F_X^b$ be the orbit containing $(x, h) \in \mathcal{E}_X^{b,U(1)}$.

$$\mathcal{O}(x, h) = \{\sigma_u^b(x, h) : u \in G\} = \{(t_u(x), \pi_b(u)h) : u \in G\}.$$

By Lemma 3.4, any element of $\mathcal{O}(x, h)$ is written as $(y_{h'}, h')$, where $y_{h'}$ is an element of $\mathcal{E}_X^b$ determined by $h' \in S(\mathcal{H}_b)$ uniquely. Hence F_X^b is a family of spheres in $\mathcal{E}_X^{b,U(1)}$, each homeomorphic to $S(\mathcal{H}_b)$.

Lemma 3.6. *For (y, h') in $\mathcal{O}(x, h)$, if $y = x \neq 0$, then $h = h'$.*

Proof. By the choice of (x, h'), there is $u \in G$ such that $\sigma_u^b(x, h) = (x, h')$. $t_u^b(x) = x$ and $\pi_b(u)h = h'$. Since $\mu_b(h') = \Pi_X^b(x) = \mu_b(h)$, there is $c \in U(1)$ such that $h' = ch$. Hence we can choose $u = cI$. Then we have

$$x = t_u^b(x) = t_{cI}^b(x) = \bar{c}t_I^b(x) = \bar{c}x,$$

by (3.5). Therefore $c = 1$ and we obtain $h = h'$ when $x \neq 0$. □

Corollary 3.7. *For $c \in U(1)$, $\mathcal{O}(x, ch) = \mathcal{O}(cx, h)$.*

Furthermore $\mathcal{O}(0, h) = \{(0, h') : h' \in S(\mathcal{H}_b)\}$. Let $(y, h') \in \mathcal{O}(x, h) \cap (\mathcal{E}_{X,\mu_b(h)} \times S(\mathcal{H}_b))$. Then there is $u \in G$ such that $(y, h') = \sigma_u(x, h)$. By the choice of (y, h'), $h' \in \mu_b^{-1}(\mu_b(h))$. Hence there is $c \in U(1)$ such that $h' = ch$.

Proposition 3.8. *F_X^b is naturally identified with the Hilbert space $\mathcal{E}_{X,\rho}$, for each $\rho \in \mathcal{P}_b$.*

3.3. Structure of the atomic bundle

We shall prove that the atomic bundle has a Hilbert bundle structure. Let $(S(\mathcal{H}_b) \times_{U(1)} F_X^b, \pi_{F_X^b}, \mathcal{P}(\mathcal{H}_b))$ be the associated bundle of $(S(\mathcal{H}_b), \mu_b, \mathcal{P}(\mathcal{H}_b))$ by F_X^b where the Hilbert space structure on F_X^b is defined according to Proposition 3.8.

Lemma 3.9. *Any element of $S(\mathcal{H}_b) \times_{U(1)} F_X^b$ can be written as $[(h, \mathcal{O}(x, h))]$ where $\mathcal{O}(x, h) \in F_X^b$.*

Proof. By definition of the associated bundle (Appendix A.3), an element of $S(\mathcal{H}_b) \times_{U(1)} F_X^b$ is the $U(1)$-orbit $[(h, \mathcal{O}(x, k))]$. Take an element $[(h, \mathcal{O}(y, k))] \in S(\mathcal{H}_b) \times_{U(1)} F_X^b$. By definition of $\mathcal{O}(y, k)$ and the transitivity of the action of G on $S(\mathcal{H})$, there is $u \in G$ such that $h = uk$ and $(t_u^b(y), h) \in \mathcal{O}(y, k)$. Denote $x \equiv t_u(y)$. Then $\mathcal{O}(x, h) = \mathcal{O}(y, k)$. Hence $[(h, \mathcal{O}(y, k))] = [(h, \mathcal{O}(x, h))]$. □

From now on, we shall denote

$$[h, x] \equiv [(h, \mathcal{O}(x, h))] \in S(\mathcal{H}_b) \times_{U(1)} F_X^b$$

for $h \in S(\mathcal{H})$ and $x \in \mathcal{E}_X^b$.

Recall for each $b \in B$, $\mathcal{P}_b$ is a Kähler manifold which is isomorphic to a projective Hilbert space $\mathcal{P}(\mathcal{H}_b)$ by the map τ^b.

Theorem 3.10. *For each $b \in B$, the B-fiber $(\mathcal{E}_X^b, \Pi_X^b, \mathcal{P}_b)$ at b is a locally trivial Hilbert bundle isomorphic to $(S(\mathcal{H}_b) \times_{U(1)} F_X^b, \pi_{F_X^b}, \mathcal{P}(\mathcal{H}_b))$.*

Proof. Define a map $\Psi^b : \mathcal{E}_X^b \to S(\mathcal{H}_b) \times_{U(1)} F_X^b$ by $\Psi^b(x) \equiv [h_x, x] \quad (x \in \mathcal{E}_X^b)$, where $h_x \in \mu_b^{-1}(\Pi_X^b(x))$. If $\Psi^b(x) = \Psi^b(y)$ for $x, y \in \mathcal{E}_X^b$, then $[h, x] = [h', y]$. Therefore there is $c \in U(1)$ such that $(h, \mathcal{O}(x, h))c = (h', \mathcal{O}(y, h'))$. By $h' = \bar{c}h$ and Corollary 3.7, $\mathcal{O}(y, \bar{c}h) = \mathcal{O}(x, \bar{c}h)$. By Lemma 3.4, we get $x = y$, so that Ψ^b is injective. By definition of F_X^b, Ψ^b is surjective, hence Ψ^b is a bijection. We obtain a set-theoretical isomorphism (Ψ^b, τ^b) of fibrations between $(\mathcal{E}_X^b, \Pi_X^b, \mathcal{P}_b)$ and $(S(\mathcal{H}_b) \times_{U(1)} F_X^b, \pi_{F_X^b}, \mathcal{P}_b)$ such that any restriction $\Psi^b|_{\mathcal{E}_{X,\rho}}$ of Ψ^b at a fiber $\mathcal{E}_{X,\rho}$ is a unitary between $\mathcal{E}_{X,\rho}$ and $\pi_{F_X^b}^{-1}(\rho)$ for $\rho \in \mathcal{P}_b$.

This isomorphism induces on $(\mathcal{E}_X^b, \Pi_X^b, \mathcal{P}_b)$ the Hilbert bundle structure of the associated bundle $(S(\mathcal{H}_b) \times_{U(1)} F_X^b, \pi_{F_X^b}, \mathcal{P}_b)$. □

By Theorem 3.10 and Definition 3.2, we have constructed in a canonic way a locally trivial Hilbert bundle from a Hilbert C^*-module and we understand that the atomic bundle of a Hilbert C^*-module is a family of associated bundles of Hopf bundles indexed by spectrum B:

$$\mathcal{E}_X \cong \bigcup_{b \in B} (S(\mathcal{H}_b) \times_{U(1)} F_X^b).$$

We conclude this section by introducing a bundle map to be used in Section 5. Let $(X \times \mathcal{P}, t, \mathcal{P})$ be the trivial complex vector bundle on $\mathcal{P}$. Then we introduce the map

$$P_X : X \times \mathcal{P} \to \mathcal{E}_X; \quad P_X(\xi, \rho) \equiv [\xi]_\rho \quad ((\xi, \rho) \in X \times \mathcal{P}). \tag{3.7}$$

The image $P_X(X \times \mathcal{P}_b)$ is dense in $\mathcal{E}_X^b$, for every $b \in B$. Moreover $P_X(\xi', \rho) = P_X(\xi, \rho)$ if and only if $\xi' - \xi \in N_\rho$, for $\rho \in \mathcal{P}$. Then (P_X, id) is a bundle map from $(X \times \mathcal{P}, t, \mathcal{P})$ to $(\mathcal{E}_X, \Pi_X, \mathcal{P})$.

4. Connection and $*$-action

In this section, we define a flat connection D on the atomic bundle and prove a relation between the associativity of $*$-action defined by D and the flatness of D.

4.1. The atomic connection of the atomic bundle

To define the $*$-action of $(C^\infty(\mathcal{P}), *)$ on the smooth sections of the atomic bundle of a Hilbert C^*-module X, we define a connection D of $\mathcal{E}_X$, called the atomic connection.

Let $\mathcal{E}_X = (\mathcal{E}_X, \Pi_X, \mathcal{P})$ be the atomic bundle of a Hilbert C^*-module X over a C^*-algebra $\mathcal{A}$. Let $\Gamma(\mathcal{E}_X)$ be the set of all bounded sections of $\mathcal{E}_X$, that is, $\Gamma(\mathcal{E}_X) \ni s : \mathcal{P} \to \mathcal{E}_X$ is a right inverse of Π_X and satisfies

$$\|s\| \equiv \sup_{\rho\in\mathcal{P}} \|s(\rho)\|_\rho < \infty. \tag{4.1}$$

By standard operations, $\Gamma(\mathcal{E}_X)$ is a complex linear space, isometric to the Banach direct sum $\oplus_{\rho\in\mathcal{P}}\mathcal{E}_{X,\rho}$. By Theorem 3.10, we can consider the differentiability of $s \in \Gamma(\mathcal{E}_X)$ at each B-fiber $s|_{\mathcal{P}_b} : \mathcal{P}_b \to \mathcal{E}_X^b$ for each $b \in B$ in the sense of Fréchet differentiability of Hilbert manifolds. Define $\Gamma_\infty(\mathcal{E}_X)$ the set of all B-fiberwise smooth sections in $\Gamma(\mathcal{E}_X)$.

A hermitian metric H is defined on $\mathcal{E}_X$ by

$$H_\rho(s, s') \equiv \langle s(\rho)|s'(\rho)\rangle_\rho \tag{4.2}$$

for $\rho \in \mathcal{P}$, $s, s' \in \Gamma_\infty(\mathcal{E}_X)$ [9]. Let $\mathfrak{X}(\mathcal{P})$ be the set of all B-fiberwise smooth vector fields of $\mathcal{P}$.

Definition 4.1. A connection on $\mathcal{E}_X$ is a $\mathbf{C}$-bilinear map D which is $C^\infty(\mathcal{P})$-linear with respect to $\mathfrak{X}(\mathcal{P})$ and satisfies the Leibniz law with respect to $\Gamma_\infty(\mathcal{E}_X)$:

$$D_Y(s\cdot l) = \partial_Y l \cdot s + l \cdot D_Y s$$

for $s \in \Gamma_\infty(\mathcal{E}_X)$, $l \in C^\infty(\mathcal{P})$ and $Y \in \mathfrak{X}(\mathcal{P})$.

For $h \in S(\mathcal{H}_b)$ we consider a trivializing neighborhood $\mathcal{V}_h$ for the Hopf bundle (see Appendix A.1). For a fixed $\rho \in \mathcal{V}_h$ and $Y \in \mathfrak{X}(\mathcal{P}_b)$, we denote by Y_ρ^h the corresponding tangent vector at ρ in local coordinates and by the linear operator of multiplication by number

$$-\frac{1}{2}\frac{\langle \beta_h(\rho)|Y_\rho^h\rangle}{1+\|\beta_h(\rho)\|^2}.$$

Proposition 4.2. *$D_{Y,\rho}^h \equiv \partial_{Y_\rho^h} + A_{Y,\rho}^h$ gives the local expression of a flat connection D^b on $\mathcal{E}_X^b$.*

Proof. We prove the cocycle condition for the family of linear maps $A \equiv \{A^h\}_{h\in S(\mathcal{H}_b)}$, where $A^h : \mathcal{H}_b \to \mathcal{L}(F_X^b)$ is defined by $A^h(Y_\rho^h) = A_{Y,\rho}^h$. For $\rho \in \mathcal{P}_b$, choose $h, h' \in S(\mathcal{H}_b)$ such that $\rho \in \mathcal{V}_h \cap \mathcal{V}_{h'}$. The cocycle condition for A is given by formula (A.2) in Appendix A.5.

Let $z' \equiv \beta_{h'}(\rho)$, $z \equiv \beta_h(\rho)$. By a simple computation, we get

$$-2\cdot A_{X,\rho}^h = \frac{\langle z|X_\rho^h\rangle}{1+\|z\|^2} = -2\cdot A_{X,\rho}^{h'} - \frac{\langle h|X_\rho^{h'}\rangle}{\langle h|z'+h'\rangle},$$

and formula (A.2) holds. Therefore $D_{Y,\rho}^h$ gives the local expression of a connection D^b. The curvature R^b of D^b can be expressed as:

$$R_{X,Y}^b = (dA)(X,Y) + (A\wedge A)(X,Y) \quad (X, Y \in \mathfrak{X}(\mathcal{P}_b)).$$

Since A is scalar, $A \wedge A = 0$. In a chart $(\mathcal{V}_h, \beta_h, \mathcal{H}_h)$ of $\rho \in \mathcal{P}_b$ and $z = \beta_h(\rho) \in \mathcal{H}_h$, we have

$$(d_z A)(X, Y) = XA^h_{Y,z} - YA^h_{X,z} - A^h_{[X,Y],z}.$$

By simple computation we obtain $(d_z A)(X, Y) = 0$, so that $R^b = 0$ and D^b is flat. □

Definition 4.3. We call the connection in Proposition 4.2 the atomic connection of the atomic bundle.

4.2. *The $*$-action of a function algebra on sections of the atomic bundle*

By (2.3), the function space $C^\infty(\mathcal{P})$ is a $*$-algebra with $*$-product which is generally not associative. We define the $*$-action of $(C^\infty(\mathcal{P}), *)$ on the smooth sections of the atomic bundle of a Hilbert C^*-module by using the atomic connection D of $\mathcal{E}_X$. We characterize algebraic properties, commutativity, associativity, of $*$-action by D and the curvature of $\mathcal{E}_X$ with respect to D. Now we denote by D any connection on $\mathcal{E}_X$.

Definition 4.4. We define the (right) $*$-action of $C^\infty(\mathcal{P})$ on $\Gamma_\infty(\mathcal{E}_X)$ by

$$s * l \equiv s \cdot l + \sqrt{-1} D_{X_l} s$$

for $l \in C^\infty(\mathcal{P})$ and $s \in \Gamma_\infty(\mathcal{E}_X)$, where X_l is the holomorphic Hamiltonian vector field of l with respect to the Kähler form of $\mathcal{P}$.

We give a geometric characterization of the above $*$-action.

Lemma 4.5. *For each $s \in \Gamma_\infty(\mathcal{E}_X)$ and $l, m \in C^\infty(\mathcal{P})$, the following equations hold*:

$$(s * l) * m - (s * m) * l = (\sqrt{-1}\{l, m\} + [D_{X_l}, D_{X_m}])s,$$
$$s * (l * m) - s * (m * l) = (\sqrt{-1}\{l, m\} + D_{[X_l, X_m]})s.$$

Proof. The first equation follows immediately by

$$\begin{aligned}(s * l) * m - (s * m) * l \\ &= \sqrt{-1} D_{X_m}(s \cdot l) + \sqrt{-1}((D_{X_l} s) \cdot m + \sqrt{-1} D_{X_m} D_{X_l} s) - \sqrt{-1} D_{X_l}(s \cdot m) \\ &\quad - \sqrt{-1}((D_{X_m} s) \cdot l + \sqrt{-1} D_{X_l} D_{X_m} s),\end{aligned}$$

since

$$X_m l - X_l m = \{l, m\}.$$

The proof of the second equation is analogous. □

Remark 4.6.

(i) By Lemma 4.5, the non-commutativity of $*$-action of $C^\infty(\mathcal{P})$ on $\Gamma_\infty(\mathcal{E}_X)$ depends on the connection D of $\mathcal{E}_X$ and the Kähler form of $\mathcal{P}$.

(ii) We denote three different notions, $*$-action, $*$-product and involution by the same symbol "$*$" according to custom style.

Let *the associator* $a(l,m)$ *of* $l,m \in C^\infty(\mathcal{P})$ be an operator

$$a(l,m) : \Gamma_\infty(\mathcal{E}_X) \to \Gamma_\infty(\mathcal{E}_X),$$

$$a(l,m)s \equiv (s * l) * m - s * (l * m) \quad (s \in \Gamma_\infty(\mathcal{E}_X)).$$

Then we have a relation between associativity and curvature.

Proposition 4.7. *On* $\Gamma_\infty(\mathcal{E}_X)$ *and for* $l,m \in C^\infty(\mathcal{P})$, *the following equation holds*:

$$a(l,m) - a(m,l) = R_{X_l, X_m},$$

where R is the curvature of $\mathcal{E}_X$ *with respect to D defined by*

$$R_{Y,Z} \equiv [D_Y, D_Z] - D_{[Y,Z]} \quad (Y, Z \in \mathfrak{X}(\mathcal{P})).$$

5. A sectional representation of Hilbert C^*-modules

Let us summarize our notations. Let X be a Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$, $\mathcal{K}_u(\mathcal{P})$ the image of the Gel'fand representation of $\mathcal{A}$ and $\mathcal{E}_X = (\mathcal{E}_X, \Pi_X, \mathcal{P})$ the atomic bundle of X. For the map P_X defined in (3.7), define a linear map

$$(P_X)_* : \Gamma(X \times \mathcal{P}) \to \Gamma(\mathcal{E}_X), \quad ((P_X)_*(s))(\rho) \equiv P_X(s(\rho), \rho)$$
$$(s \in \Gamma(X \times \mathcal{P}), \rho \in \mathcal{P}).$$

We define a subspace Γ_X of $\Gamma(\mathcal{E}_X)$ as follows.

Definition 5.1.

$$\Gamma_X \equiv (P_X)_*(\Gamma_{\text{const}}(X \times \mathcal{P})),$$

where $\Gamma_{\text{const}}(X \times \mathcal{P})$ is the subspace of $\Gamma(X \times \mathcal{P})$ consisting of all constant sections.

Remark 5.2. Γ_X is quite smaller that the set of all holomorphic sections of $\mathcal{E}_X$. Actually, we shall see in Theorem 5.6 that the hermitian form, restricted to Γ_X has values in $\mathcal{K}_u(\mathcal{P})$.

We prepare some lemmata for the proof of the reconstruction theorem and explain how the structure of Hilbert C^*-module is interpreted as the geometrical structure of the atomic bundle.

For $\xi \in X$, we define a section $s_\xi \in \Gamma(\mathcal{E}_X)$ of $\mathcal{E}_X$ by

$$s_\xi(\rho) \equiv [\xi]_\rho \quad (\rho \in \mathcal{P}).$$

Define the map

$$\Psi : X \to \Gamma(\mathcal{E}_X), \qquad \Psi(\xi) \equiv s_\xi \quad (\xi \in X).$$

Lemma 5.3. *The map Ψ is well defined and has the following properties*:

(i) *Ψ is a linear isometry,*
(ii) *for each $\xi \in X$, $\Psi(\xi) \in \Gamma_\infty(\mathcal{E}_X)$ and is holomorphic,*
(iii) *$\Psi(\xi) * f_A = \Psi(\xi \cdot A)$ for $\xi \in X$* and *$A \in \mathcal{A}$.*

Proof. For every $x \in X$, we have

$$\|s_\xi\| = \sup_{\rho\in\mathcal{P}} |\rho(\langle\xi|\xi\rangle)|^{1/2} = \|\langle\xi|\xi\rangle\|^{1/2} = \|\xi\|.$$

Hence s_ξ is bounded on $\mathcal{P}$ and the map Ψ well defined and isometric. This proves (i).

(ii) Let $\rho \in \mathcal{P}$, so that $\rho \in \mathcal{P}_b$ for some $b \in B$. Choose as a representative for b an irreducible representation $(\mathcal{H}, \pi)$. Fix $h \in S(\mathcal{H})$ and, using the notations in (A.1), take a local trivialization $\psi_{\alpha,h}$ of the Hopf bundle at $(\mathcal{V}_h, \beta_h, \mathcal{H}_h)$ with $\rho \in \mathcal{V}_h$. By formula (B.2) we obtain

$$\partial_Y \phi_h(\rho)(s_\xi(\rho)) = \mathcal{O}\left(\left[\partial_Y \hat{\xi}_\rho + \xi_\rho\left(K^h_{Y,\rho} - \frac{\langle z|Y\rangle}{2(1+\|z\|^2)}\right)\right]_\rho, h\right). \tag{5.1}$$

Owing to (B.3), the right-hand side of (5.1) is smooth with respect to $z \equiv \beta_h(\rho) \in \mathcal{H}_h$, and hence, s_ξ is smooth at $\mathcal{P}_b$ for each $b \in B$. For $\rho_0 \in \mathcal{P}_b$, we can choose $h_0 \in S(\mathcal{H}_b)$ such that

$$\rho_0 = \langle h_0|\pi_b(\cdot)h_0\rangle.$$

Then $\beta_{h_0}(\rho_0) = 0$. According to the proof of Lemma B.1, we have

$$\langle e|\phi_{h_0}(\rho)(s_\xi(\rho))\rangle = \frac{\langle \Omega^{h_0}_{\rho'}|\pi_b(\langle\xi'|\xi\rangle)(z+h_0)\rangle}{\sqrt{1+\|z\|^2}}$$

for $z = \beta_{h_0}(\rho)$, $\rho \in \mathcal{V}_{h_0}$. For an anti-holomorphic tangent vector $\bar{Y}$ of $\mathcal{P}_b$, we have

$$\bar{\partial}_{\bar{Y}} \phi_h(\rho)(s_\xi(\rho)) = \mathcal{O}\left(\left[-\xi_\rho \frac{\langle Y|z\rangle}{2(1+\|z\|^2)}\right]_\rho, h\right),$$

from which follows:

$$\bar{\partial}_{\bar{Y}} \phi_h(\rho)(s_\xi(\rho))|_{z=0} = 0.$$

We see that the anti-holomorphic derivative of s_ξ vanishes at each point in $\mathcal{P}_b$. Hence s_ξ is holomorphic.

(iii) Let $A \in \mathcal{A}$. For $b \in B$ and $\rho_0 \in \mathcal{P}_b$, take a chart $(\mathcal{V}_h, \beta_h, \mathcal{H}_h)$ at ρ_0, where h is a unit vector in $\mathcal{H}$ and $(\mathcal{H}, \pi)$ is a representative of b. Then for $z \in \mathcal{H}_h$, we have

$$(f_A \circ \beta_h^{-1})(z) = \frac{\langle (z+h)|\pi(A)(z+h)\rangle}{1+\|z\|^2}.$$

Then the representation $X^h_{f_A}$ of the Hamiltonian vector field X_{f_A} at $(\mathcal{V}_h, \beta_h, \mathcal{H}_h)$ is

$$(X^h_{f_A})_z = -\sqrt{-1}(\pi(A)(z+h) - \langle h|\pi(A)(z+h)\rangle(z+h))$$

for $z \in \mathcal{H}_h$. If we take h such that $\beta_h(\rho_0) = 0$, then it holds that

$$(X^h_{f_A})_0 = -\sqrt{-1}(\pi(A)h - \langle h|\pi(A)h\rangle h).$$

The connection D satisfies

$$\langle v|(D_{X_{f_A}} s)(\rho_0)\rangle_{\rho_0} = \partial_{\rho_0}(\langle v|s(\cdot)\rangle_{\rho_0})(X_{f_A})$$

for $v \in E_h$, $s \in \Gamma_\infty(\mathcal{E}_X)$. Hence we have

$$(D_{X_{f_A}} s_\xi)(\rho_0) = [\xi a_{X_{f_A},0}]_{\rho_0},$$

where $a_{X_{f_A},0} \in \mathcal{A}$ satisfies

$$\pi(a_{X_{f_A},0})h = X_{f_A} = -\sqrt{-1}(\pi(A) - \langle h|\pi(A)h\rangle)h.$$

Therefore we have

$$\begin{aligned}\sqrt{-1}(D_{X_{f_A}} s_\xi)(\rho_0) &= \sqrt{-1}[\xi \cdot (-\sqrt{-1}(A - \langle h|\pi(A)h\rangle))]_{\rho_0} \\ &= [\xi A]_{\rho_0} - [\xi]_{\rho_0}\langle h|\pi(A)h\rangle = s_{\xi A}(\rho_0) - s_\xi(\rho_0) f_A(\rho_0),\end{aligned}$$

from which follows:

$$(s_\xi * f_A)(\rho_0) = s_\xi(\rho_0) f_A(\rho_0) + \sqrt{-1}(D_{X_{f_A}} s_\xi)(\rho_0) = s_{\xi A}(\rho_0).$$

Therefore we obtain $\Psi(\xi) * f_A = \Psi(\xi A)$. □

Proposition 5.4. *Any element in Γ_X is holomorphic.*

Proof. For each $\tau \in \Gamma_{\mathrm{const}}(X \times \mathcal{P})$, there is $\xi \in X$ such that $\tau(\rho) = (\xi, \rho)$ for $\rho \in \mathcal{P}$. Then

$$s \in \Gamma_X \Leftrightarrow s(\rho) = [\xi]_\rho \quad \text{for each } \rho \in \mathcal{P} \Leftrightarrow s = \Psi(\xi) \Leftrightarrow s \in \Psi(X).$$

Hence $\Gamma_X = \Psi(X)$. Therefore Proposition 5.4 follows from Lemma 5.3 (ii). □

Lemma 5.5.

(i) *Γ_X is a right $\mathcal{K}_u(\mathcal{P})$-module by the $*$-action defined in Definition 4.4.*

(ii) *For the hermitian metric H of $\mathcal{E}_X$, let $\mathfrak{h}$ be the restriction of H to Γ_X. Then the function-valued sesquilinear form*

$$\mathfrak{h} : \Gamma_X \times \Gamma_X \to C^\infty(\mathcal{P}),$$

satisfies

$$\begin{aligned}&\mathfrak{h}(s, s') \in \mathcal{K}_u(\mathcal{P}) \quad (s, s' \in \Gamma_X), \qquad \overline{\mathfrak{h}(s, s')} = \mathfrak{h}(s', s) \quad (s, s' \in \Gamma_X), \\ &\mathfrak{h}(s, s) \geq 0 \quad (s \in \Gamma_X), \qquad \mathfrak{h}(s, s' * f) = \mathfrak{h}(s, s') * f \quad (s, s' \in \Gamma_X,\ f \in \mathcal{K}_u(\mathcal{P})), \\ &\|\mathfrak{h}(s, s)\|^{1/2} = \|s\| \quad (s \in \Gamma_X),\end{aligned} \tag{5.2}$$

where the positivity in (5.2) means that $\mathfrak{h}(s, s)$ is a positive-valued function on $\mathcal{P}$ and the norm of $\mathfrak{h}(s, s)$ is the one defined in (4.1).

(iii) *The following equation holds*:

$$\mathfrak{h}_\rho(\Psi(\xi), \Psi(\eta)) = \rho(\langle \xi|\eta\rangle) \quad (\xi, \eta \in X, \rho \in \mathcal{P}).$$

Proof. By Proposition 5.4 we know that $\Gamma_X = \Psi(X)$. By $\mathcal{K}_u(\mathcal{P}) = f(\mathcal{A})$ and Lemma 5.3 (iii), the map is a module action. Thus (i) is verified.

(ii) and (iii): Next, we have the following equations

$$\mathfrak{h}_\rho(\Psi(\xi), \Psi(\xi')) = H_\rho(s_\xi, s_{\xi'}) = \langle s_\xi(\rho)|s_{\xi'}(\rho)\rangle_\rho = \rho(\langle \xi|\xi'\rangle),$$

which proves (iii). Furthermore, $\rho(\langle \xi|\xi'\rangle) = f_{\langle \xi|\xi'\rangle}(\rho)$. Therefore $\mathfrak{h}(\Psi(\xi), \Psi(\xi')) = f_{\langle \xi|\xi'\rangle} \in \mathcal{K}_u(\mathcal{P})$. Hence $\mathfrak{h}(s, s') \in \mathcal{K}_u(\mathcal{P})$ for each $s, s' \in \Gamma_X$. For $\xi, \eta \in X$, $A \in \mathcal{A}$,

$$\mathfrak{h}_\rho(s_\eta, s_\xi * f_A) = \mathfrak{h}_\rho(s_\eta, s_{\xi A}) = \rho(\langle \eta|\xi A\rangle) = (f_{\langle \eta|\xi\rangle} * f_A)(\rho) = (\mathfrak{h}(s_\eta, s_\xi) * f_A)(\rho) \quad (\text{by using (iii)}).$$

Hence $\mathfrak{h}(s, s' * l) = \mathfrak{h}(s, s') * l$ for $s, s' \in \Gamma_X, l \in \mathcal{K}_u(\mathcal{P})$. The other equations in statement (ii) follow from the property of the C^*-valued inner product of X and by the proof of Lemma 5.3 (i). □

Finally we come to the reconstruction theorem for Hilbert C^*-modules by means of their atomic bundle.

Theorem 5.6.

(i) *Any element in Γ_X is holomorphic.*
(ii) *Γ_X is a Hilbert C^*-module over the C^*-algebra $\mathcal{K}_u(\mathcal{P})$.*
(iii) *There is a Banach space isomorphism $\Psi : X \to \Gamma_X$*

$$\begin{array}{ccc} X \times \mathcal{A} & \rightarrow & X \\ {\scriptstyle \Psi\times f}\downarrow & & \downarrow{\scriptstyle \Psi} \\ \Gamma_X \times \mathcal{K}_u(\mathcal{P}) & \rightarrow & \Gamma_X, \end{array}$$

where the horizontal arrows are module actions. Hence, under the identification $f : \mathcal{A} \cong \mathcal{K}_u(\mathcal{P})$, Γ_X is isomorphic to X as a Hilbert $\mathcal{A}$-module.

Proof. (i) is Proposition 5.4. (ii) By Lemma 5.5 (i), (ii) and Definition 3.1

$$\mathfrak{h} : \Gamma_X \times \Gamma_X \to \mathcal{K}_u(\mathcal{P}), \tag{5.3}$$

is a positive definite C^*-inner product of a right $\mathcal{K}_u(\mathcal{P})$-module Γ_X. Hence Γ_X is a Hilbert C^*-module over a C^*-algebra $\mathcal{K}_u(\mathcal{P})$.

(iii) By Lemma 5.3 (i) and Proposition 5.4, Ψ is an isomorphism between X and Γ_X. If we rewrite the module actions ϕ and ψ of X and Γ_X, respectively, by

$$\phi(\xi, A) = \xi A, \qquad \psi(s, l) = s * l$$

for $\xi \in X$, $A \in \mathcal{A}$, $s \in \Gamma_X$ and $l \in \mathcal{K}_u(\mathcal{P})$, then we have

$$(\psi \circ (\Psi \times f))(\xi, A) = \Psi(\xi) * f_A = s_{\xi A} = (\Psi \circ \phi)(\xi, A),$$

by Lemma 5.3 (iii). Hence we obtain:

$$\psi \circ (\Psi \times f) = \Psi \circ \phi.$$

Therefore the diagram in the statement (iii) is commutative. □

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Appendix A. The Hopf bundle over a Hilbert space

We recall some facts about the Hopf bundle over a Hilbert space $\mathcal{H}$ and its associated bundle. Note we do *not* assume $\dim \mathcal{H} < \infty$.

A.1. Definition

We denote $\mathcal{H}$ a Hilbert space over $\mathbf{C}$ with $\dim \mathcal{H} \geq 1$. Define

$$S(\mathcal{H}) \equiv \{z \in \mathcal{H} : \|z\| = 1\}, \qquad \mathcal{P}(\mathcal{H}) \equiv (\mathcal{H} \setminus \{0\})/\mathbf{C}^{\times}.$$

We call $S(\mathcal{H})$ and $\mathcal{P}(\mathcal{H})$ *a Hilbert sphere* and *a projective Hilbert space* over $\mathcal{H}$, respectively. We denote an element of $\mathcal{P}(\mathcal{H})$ by $[z]$ for $z \in \mathcal{H} \setminus \{0\}$. It is well known that $S(\mathcal{H})$ is a submanifold of $\mathcal{H}$, in the relative topology. We give $\mathcal{P}(\mathcal{H})$ the quotient topology from $\mathcal{H} \setminus \{0\} \subset \mathcal{H}$ by the natural projection. Define a projection μ from $S(\mathcal{H})$ to $\mathcal{P}(\mathcal{H})$ by

$$\mu : S(\mathcal{H}) \to \mathcal{P}(\mathcal{H}), \qquad \mu(z) \equiv [z] \quad (z \in S(\mathcal{H})).$$

We call $(S(\mathcal{H}), \mu, \mathcal{P}(\mathcal{H}))$ *the Hopf* (*fiber*) *bundle over* $\mathcal{H}$. Clearly, $\mu^{-1}([z]) \cong S^1$ for each $[z] \in \mathcal{P}(\mathcal{H})$.

We define local trivial neighborhoods of the Hopf bundle [3]. Fix $h \in S(\mathcal{H})$ and define

$$\mathcal{V}_h \equiv \{[z] \in \mathcal{P}(\mathcal{H}) : \langle h|z\rangle \neq 0\}, \qquad \mathcal{H}_h \equiv \{z \in \mathcal{H} : \langle h|z\rangle = 0\},$$
$$\beta_h : \mathcal{V}_h \to \mathcal{H}_h, \qquad \beta_h([z]) \equiv \frac{z}{\langle h|z\rangle} - h \qquad ([z] \in \mathcal{V}_h).$$

Then $\{(\mathcal{V}_h, \beta_h, \mathcal{H}_h)\}_{h \in S(\mathcal{H})}$ is a holomorphic atlas for $\mathcal{P}(\mathcal{H})$. As well known, $\mathcal{P}(\mathcal{H})$ is a Kähler manifold [3].

Let ψ_h be the local trivializing neighborhoods for $S(\mathcal{H})$ at $\mathcal{V}_h$ defined by

$$\psi_h : \mu^{-1}(\mathcal{V}_h) \cong \mathcal{V}_h \times U(1), \qquad \psi_h(z) \equiv ([z], \phi_h(z)),$$
$$\phi_h(z) \equiv \frac{\langle z|h\rangle}{|\langle h|z\rangle|} \quad (z \in \mu^{-1}(\mathcal{V}_h)),$$
$$\psi_h^{-1}([z], g) \equiv z\frac{\langle h|z\rangle}{|\langle h|z\rangle|}g \qquad ([z] \in \mathcal{V}_h, g \in U(1)).$$

Hence $\{\mathcal{V}_h\}_{h\in S(\mathcal{H})}$ is a system of local trivial neighborhoods for $(S(\mathcal{H}), \mu, \mathcal{P}(\mathcal{H}))$. Let R be the right action of $U(1)$ on $S(\mathcal{H})$ defined by

$$S(\mathcal{H}) \times U(1) \to S(\mathcal{H}); \qquad (z, c) \mapsto z \cdot c = R_c z \equiv \bar{c}z.$$

Then the following conditions are satisfied: (i) $\mu(R_c z) = \mu(z)$, (ii) R is free, that is, if $R_c z = z$, then $c = 1$, (iii) for each $h \in S(\mathcal{H})$:

$$\phi_h(R_c z) = \frac{\langle z|h\rangle}{|\langle h|z\rangle|}c \quad (z \in S(\mathcal{H}), c \in U(1)).$$

Hence $(S(\mathcal{H}), \mu, \mathcal{P}(\mathcal{H}))$ is a principal $U(1)$-bundle.

Lemma A.1. *Let $h, h' \in S(\mathcal{H})$ with $\mathcal{V}_{h'} \cap \mathcal{V}_h \neq \emptyset$. For $z, X \in \mathcal{H}_h$, we have*

$$(\beta_{h'} \circ \beta_h^{-1})(z) = \frac{h+z}{\langle h'|h+z\rangle} - h',$$
$$\partial_z(\beta_{h'} \circ \beta_h^{-1})(X) = \frac{1}{\langle h'|h+z\rangle}X - \frac{\langle h'|X\rangle}{\langle h'|h+z\rangle^2}(h+z).$$

Definition A.2. We denote by Ω_h the local section

$$\Omega_h([z]) \equiv \phi_h(z)z \quad ([z] \in \mathcal{V}_h).$$

By definition, $\langle h|\Omega_h(\rho)\rangle > 0$ for $\rho \in \mathcal{V}_h$.

A.2. Transition functions

Let $h, h' \in S(\mathcal{H})$ with $h' \in \mathcal{V}_h$, then the transition function $Q_{h'h} : \mathcal{V}_h \cap \mathcal{V}_{h'} \to U(1)$ is

$$Q_{h'h}([z]) \equiv \frac{\langle z|h'\rangle}{|\langle h'|z\rangle|}\frac{\langle h|z\rangle}{|\langle h|z\rangle|}.$$

Fact A.3.

(i) $Q_{hh}([z]) = 1$ for $[z] \in \mathcal{V}_h$.
(ii) If $h, h' \in S(\mathcal{H})$ satisfy $\langle h'|h\rangle \neq 0$, then $Q_{h'h} = Q_{hh'}^{-1}$.
(iii) If $h, h', h'' \in S(\mathcal{H})$ are mutually non-orthogonal, then

$$Q_{h''h'}([z]) \cdot Q_{h'h}([z]) = Q_{h''h}([z]) \quad ([z] \in \mathcal{V}_h \cap \mathcal{V}_{h'} \cap \mathcal{V}_{h''}).$$

Lemma A.4. *Let X be a tangent vector of $\mathcal{P}(\mathcal{H})$ at $\rho \in \mathcal{V}_h \cap \mathcal{V}_{h'}$ which is realized in $\mathcal{H}_{h'}$ and $\beta_{h'}(\rho) = z$. Then*

$$\partial_z(Q_{h'h}^{-1} \circ \beta_{h'}^{-1})(X) = -\frac{1}{2}\frac{\langle z + h'|h\rangle^2 \langle h|X\rangle}{|\langle h|z + h'\rangle|^3}.$$

Proof. The statement is proved by a simple computation. □

Lemma A.5. *In the notation of* Lemma A.4 we have:

$$(Q_{h'h} \circ \beta_{h'}^{-1})(w) \cdot \partial_w(Q_{h'h}^{-1} \circ \beta_{h'}^{-1})(X) = -\frac{1}{2}\frac{\langle h|X\rangle}{\langle h|w + h'\rangle}.$$

Proof. By Lemma A.4 and the definition of Q, the statement follows easily. □

A.3. Associated bundles of Hopf bundles

Let F be a C^∞-manifold with a left $U(1)$-action α and $S(\mathcal{H}) \times F$ the direct product space of $S(\mathcal{H})$ and F. Define a right $U(1)$-action γ on $S(\mathcal{H})$ by

$$z\gamma_c \equiv \bar{c}z \quad (c \in U(1), z \in S(\mathcal{H})).$$

We define $S(\mathcal{H}) \times_{U(1)} F$ as the set of all $U(1)$-orbits in $S(\mathcal{H}) \times F$, where the $U(1)$-action is defined by

$$(z, f)c \equiv (z\gamma_c, \alpha(\bar{c}) f) \qquad (c \in U(1), (z, f) \in S(\mathcal{H}) \times F).$$

The topology of $S(\mathcal{H}) \times_{U(1)} F$ is induced from $S(\mathcal{H}) \times F$ by the natural projection $\pi : S(\mathcal{H}) \times F \to S(\mathcal{H}) \times_{U(1)} F$. We denote the element of $S(\mathcal{H}) \times_{U(1)} F$ containing (x, f) by $[(x, f)]$. Define a projection

$$\pi_F : S(\mathcal{H}) \times_{U(1)} F \to \mathcal{P}(\mathcal{H}), \qquad \pi_F([(x, f)]) \equiv \mu(x).$$

The fibration $\mathbf{F} \equiv (S(\mathcal{H}) \times_{U(1)} F, \pi_F, \mathcal{P}(\mathcal{H}))$ is called *the associated bundle of* $(S(\mathcal{H}), \mu, \mathcal{P}(\mathcal{H}))$ *by* F. For $h \in S(\mathcal{H})$, define a map $\psi_{\alpha,h} : \pi_F^{-1}(\mathcal{V}_h) \to \mathcal{V}_h \times F$ by

$$\psi_{\alpha,h}([(z, f)]) \equiv (\mu(z), \phi_{\alpha,h}([(z, f)])) = (\mu(z), \alpha(\phi_h(z)) f). \tag{A.1}$$

We have

$$\psi_{\alpha,h}^{-1}([z], f) = \left[\left(z, \alpha\left(\frac{\langle h|z\rangle}{|\langle h|z\rangle|}\right) f\right)\right] \quad (([z], f) \in \mathcal{V}_h \times F).$$

The definition of $\psi_{\alpha,h}$ is independent of the choice of (z, f). Hence $\psi_{\alpha,h}$ is a local trivialization of $\mathbf{F}$ at $\mathcal{V}_h$. The transition function on $\mathcal{V}_h \cap \mathcal{V}_{h'} \times F$ is therefore

$$\hat{Q}_{\alpha,h',h} \equiv \psi_{\alpha,h'} \circ \psi_{\alpha,h}^{-1} : ([z], f) \mapsto ([z], \alpha(Q_{h'h}([z])) f).$$

If F is a complex vector space, then the natural $U(1)$-action α is the scalar multiplication.

A.4. Recovery of the typical fiber

Let $(S(\mathcal{H}), \mu, \mathcal{P}(\mathcal{H}))$ be a Hopf bundle and F a complex Hilbert space. We consider the associated bundle $S(\mathcal{H}) \times_{U(1)} F$ which is done w.r.t. the natural action of $U(1)$ on F.

Proposition A.6. *There is the following equivalence of fiber bundles on* $\mathcal{P}(\mathcal{H})$:

$$(S(\mathcal{H}) \times_{U(1)} F) \times_{\mathcal{P}(\mathcal{H})} S(\mathcal{H}) \cong S(\mathcal{H}) \times F.$$

Proof. Let X_1 be the l.h.s. in the above statement. We note that any element of X_1 is written as $([(h, v)], h)$, where $[(h, v)] \in S(\mathcal{H}) \times_{U(1)} F$ because $\pi_F([(h, v)]) = \mu(h)$ and we can choose the phase factor of (h, v) according to h. Let

$$\hat{\pi}_F : X_1 \to \mathcal{P}(\mathcal{H}), \qquad \hat{\pi}_F([(h, v)], h) \equiv h.$$

Define

$$\Phi : X_1 \to S(\mathcal{H}) \times F, \qquad \Phi([(h, v)], h) \equiv (h, v).$$

Then Φ is well defined and bijective. Furthermore

$$(\mu_F \circ \Phi)([(h, v)], h) = \mu_F(h, v) = h = \hat{\pi}_F([(h, v)], h).$$

Therefore $\mu_F \circ \Phi = \hat{\pi}_F$ and (Φ, id) is a bundle map between X_1 and $(S(\mathcal{H}) \times F, \mu_F, \mathcal{P}(\mathcal{H}))$. □

Proposition A.7. *Let α be a transitive action of a group G on $S(\mathcal{H})$ and consider the action*

$$\hat{\alpha} \equiv (\alpha \times_{U(1)} 1) \times_{\mathcal{P}(\mathcal{H})} \alpha,$$

of G on $X_1 \equiv (S(\mathcal{H}) \times_{U(1)} F) \times_{\mathcal{P}(\mathcal{H})} S(\mathcal{H})$. Then the quotient Y_1 of X_1 w.r.t. the action $\hat{\alpha}$ is canonically identified with the linear space F.

Proof. For $[x] = [([(h, v)], h)] \in Y_1$, $[x] = \{([(\alpha_g h, v)], \alpha_g h) : g \in G\}$. Hence we can take the quotient Φ to the bijection

$$\tilde{\Phi} : Y_1 \to S(\mathcal{H}) \times F, \qquad \tilde{\Phi}([x]) \equiv [\Phi(x)].$$ □

A.5. Connections on an associated bundle of a Hopf bundle

Let $\mathbf{F} \equiv (S(\mathcal{H}) \times_{U(1)} F, \pi_F, \mathcal{P}(\mathcal{H}))$ be an associated vector bundle of a Hopf bundle $(S(\mathcal{H}), \mu, \mathcal{P}(\mathcal{H}))$ by a complex Hilbert space F. Let $\Gamma(\mathbf{F})$ be the set of all smooth sections of $\mathbf{F}$, that is the set of right inverses of the projection π_F. By the standard operations, $\Gamma(\mathbf{F})$ is a complex linear space.

Definition A.8. D is connection on $\mathbf{F}$ if D is a bilinear map of complex vector spaces $D : \mathfrak{X}(\mathcal{P}(\mathcal{H})) \times \Gamma(\mathbf{F}) \to \Gamma(\mathbf{F})$ which is $C^\infty(\mathcal{P}(\mathcal{H}))$-linear with respect to $\mathfrak{X}(\mathcal{P}(\mathcal{H}))$ and satisfies the Leibniz law with respect to $\Gamma(\mathbf{F})$:

$$D_Y(s \cdot l) = \partial_Y l \cdot s + l \cdot D_Y s \quad (s \in \Gamma(\mathbf{F}), l \in C^\infty(\mathcal{P}(\mathcal{H}))Y \in \mathfrak{X}(\mathcal{P}(\mathcal{H}))).$$

For $Y \in \mathfrak{X}(\mathcal{P}(\mathcal{H}))$, $h \in S(\mathcal{H})$ and $\rho \in \mathcal{V}_h$, we denote by Y^h_ρ the corresponding tangent vector at ρ in a local chart. Assume that a connection D on $\mathbf{F}$ can be written as $\partial + A$. Then $D_Y = \partial_Y + A_Y$ is a linear map on $\Gamma(\mathbf{F})$. If $D_Y|^h_\rho$ is the local expression of D_Y on the local trivial chart w.r.t. $h \in S(\mathcal{H})$ and $\rho \in \mathcal{V}_h$, then we obtain families of linear maps

$$A^h_{Y,\rho} : F \to F,$$

such that $\partial_Y|^h_\rho + A^h_{Y,\rho} = (\partial_Y + A_Y)^h_\rho = (\partial + A)^h_{Y,\rho}$.

Fact A.9. The family $\{A^h_{Y,\rho}\}$ is the local expression of a connection $D \equiv \partial + A$ on $\mathbf{F}$ if and only if the following equality is satisfied:

$$A^{h'}_{Y,\rho} = -\frac{1}{2}\frac{\langle h|Y\rangle}{\langle h|z+h'\rangle} + A^h_{Y,\rho} \quad (\rho \in \mathcal{V}_{h'} \cap \mathcal{V}_h), \tag{A.2}$$

where Y is a holomorphic tangent vector of $\mathcal{P}(\mathcal{H})$ at ρ which is realized on $\mathcal{H}_{h'}$ and $z = \beta_{h'}(\rho)$.

Proof. By Leibniz rule and Lemma A.5, we obtain the formula directly. □

Appendix B. Lemma for the main theorem

We prepare some equations for the main theorem. For $\rho \in \mathcal{V}_h$, define a vector in Ω^h_ρ in $\mathcal{H}_b$ by

$$\Omega^h_\rho \equiv \frac{\beta_h(\rho) + h}{\sqrt{1 + \|\beta_h(\rho)\|^2}}.$$

Assume that $\rho = \omega_x \circ \pi_b$ for $x \in \mathcal{H}_b$, $\|x\| = 1$. Then $[x] = [\Omega^h_\rho]$ and $\langle h|\Omega^h_\rho\rangle > 0$. Let s be a section in $\Gamma(\mathcal{E}_X)$ such that for each $\rho \in \mathcal{P}_b$, there is $\xi_\rho \in X$ which satisfies $s(\rho) = [\xi_\rho]_\rho \in \mathcal{E}_{X,\rho}$. Let $z = \beta_h(\rho)$ for $h \in S(\mathcal{H}_b)$ such that $\rho \in \mathcal{V}_h$.

Lemma B.1. *Let $\psi_{\alpha,h}$ be as in (A.1). Then the following equations hold*:

$$\langle e|\psi_{\alpha,h}(s(\rho))\rangle = \frac{\langle \Omega^h_{\rho'}|\pi_b(\langle \xi'|\xi_\rho\rangle)(z+h)\rangle}{\sqrt{1+\|z\|^2}} (e = \mathcal{O}([\xi']_{\rho'}, h) \in F^b_X), \tag{B.1}$$

$$\partial_Y \phi_h(\rho)(s(\rho)) = \mathcal{O}\left(\left[\partial_Y \hat{\xi}_\rho + \xi_\rho\left(K^h_{Y,\rho} - \frac{\langle z|Y\rangle}{2(1+\|z\|^2)}\right)\right]_\rho, h\right), \tag{B.2}$$

where $K^h_{Y,\rho} \in \mathcal{A}$ is defined by

$$\pi_b(K^h_{Y,\rho})(h+z) = Y, \tag{B.3}$$

and $[\partial_Y \hat{\xi}_\rho]_\rho \in \mathcal{E}_{X,\rho}$ is defined by

$$\langle [\eta]_\rho | [\partial_Y \hat{\xi}_\rho]_\rho\rangle_\rho \equiv \rho(\partial_Y \langle \eta|\xi_\rho\rangle)$$

for $[\eta]_\rho \in \mathcal{E}_{X,\rho}$.

Proof. Let $\phi_{\alpha,h} : (\Pi_X^b)^{-1}(\mathcal{V}_h) \to F_X^b$ be the map defined by

$$\psi_{\alpha,h}(x) = (\mu_b(h), \phi_{\alpha,h}(x)).$$

For $e = \mathcal{O}([\xi']_{\rho'}, h) \in F_X^b$ such that $h \in \mu_b^{-1}(\rho)$, we have

$$\langle e|\phi_{\alpha,h}(s(\rho))\rangle = \langle \mathcal{O}([\xi']_{\rho'}, h')|\mathcal{O}([\xi]_\rho, h)\rangle = \frac{\langle \Omega_{\rho'}^h|\pi_b(\langle \xi'|\xi_\rho\rangle)(z+h)\rangle}{\sqrt{1+\|z\|^2}}.$$

From this we get

$$\langle e|\partial_Y\phi_h(\rho)(s(\rho))\rangle = \frac{\langle \Omega_{\rho'}^h|\pi_b(\partial_Y\langle \xi'|\xi_\rho\rangle)(z+h)\rangle}{\sqrt{1+\|z\|^2}} + \frac{\langle \Omega_{\rho'}^h|\pi_b(\langle \xi'|\xi_\rho\rangle)Y\rangle}{\sqrt{1+\|z\|^2}} - \frac{\langle \Omega_{\rho'}^h|\pi_b(\langle \xi'|\xi_\rho\rangle)(z+h)\rangle\langle z|Y\rangle}{2(\sqrt{1+\|z\|^2})^3}.$$

Hence we obtain

$$\partial_Y\phi_h(\rho)(s(\rho)) = \mathcal{O}\left(\left[\partial_Y\hat{\xi}_\rho + \xi_\rho\left(K_Y^h - \frac{\langle z|Y\rangle}{2(1+\|z\|^2)}\right)\right]_\rho, h\right). \qquad \square \tag{B.4}$$

References

[1] O. Bratteli, D.W. Robinson, Operator Algebras and Quantum Statistical Mechanics I, II, Springer, New York, 1979, 1981.

[2] N. Bourbaki, Elements of Mathematics, General Topology: Part I, Addison-Wesley, Reading, MA, 1966.

[3] R. Cirelli, A. Manià, L. Pizzocchero, A functional representation of noncommutative C^*-algebras, Rev. Math. Phys. 6 (5) (1994) 675–697.

[4] A. Connes, Non commutative differential geometry, Publ. Math. IHES 62 (1986) 257–360.

[5] A. Connes, Non Commutative Geometry, Academic Press, Orlando, 1993.

[6] K.K. Jensen, K. Thomsen, Elements of KK-Theory, Birkhauser, Basel, 1991.

[7] R.V. Kadison, J.R. Ringrose, Fundamentals of the Theory of Operator Algebras, vols. I–IV, Academic Press, New York, 1983.

[8] M. Karoubi, K-Theory An Introduction, Springer, Berlin, 1978.

[9] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, vol. I, Interscience, New York, 1969.

[10] G.K. Pedersen, C^*-algebras and their Automorphism Groups, Academic Press, New York, 1979.

[11] J.C. Várilly, J.M. Gracia-Bondía, Connes' noncommutative differential geometry and the standard model, J. Geom. Phys. 12 (1993) 223–301.